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# There are no hereditary productive $\gamma$-spaces

Francis Jordan

*Department of Mathematical Sciences, Box 8093, Georgia Southern University, Statesboro, GA 30458, USA*

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ABSTRACT

We show that if $X$ is an uncountable productive $\gamma$-set [F. Jordan, Productive local properties of function spaces, Topology Appl. 154 (2007) 870–883], then there is a countable $Y \subseteq X$ such that $X \setminus Y$ is not Hurewicz.

Along the way we answer a question of A. Miller by showing that an increasing countable union of $\gamma$-spaces is again a $\gamma$-space. We will also show that $\lambda$-spaces with the Hurewicz property are precisely those spaces for which every co-countable set is Hurewicz.



## 1. Introduction

Our main subject in this paper is the class of (productive) $\gamma$-spaces. The term productive $\gamma$-space is misleading because it does not mean a space whose product with every $\gamma$-space is a $\gamma$-space. While the product of a productive $\gamma$-space and a $\gamma$-space is again a $\gamma$-space, it is not known if this property characterizes productively $\gamma$-spaces.

In [6], under the Continuum Hypothesis, an uncountable productive $\gamma$-subspace of $\mathbb{R}$ was constructed. The construction was based on the construction of Galvin and Miller [2], under Martin's Axiom, of a $\gamma$-space of size continuum in $\mathbb{R}$. Both examples have the property that there is a countable subset whose removal will make the space not have the Hurewicz property [2]. There do exist uncountable $\gamma$-spaces that remain $\gamma$-spaces, and hence have the Hurewicz property, when any subset is removed, see [2]. Our main purpose is to show that this is not the case for *productive* $\gamma$-space. We will prove:

**Theorem 1.** *If $X$ is an uncountable productive $\gamma$-space, then there is a countable $Y \subseteq X$ such that $X \setminus Y$ is not Hurewicz.*

Along the way we will prove a general result about Fréchet filters to gain information about countable unions of $\gamma$-spaces and productive $\gamma$-spaces. In particular, we answer a question of Miller [11], which is again asked in [15]. We will also use methods from [16] to establish a result about $\lambda$-spaces with the Hurewicz property which is of independent interest.

## 2. Terminology

We use standard set theoretic notation. Ordinals are identified with their set of predecessors. By $\omega$ we denote the first infinite ordinal. For a set $X$ we denote the finite and countably infinite subsets of $X$ by $[X]^{<\omega}$ and $[X]^{\omega}$, respectively. Given sets $X$ and $Y$ we denote the set of all functions with domain $X$ and range contained in $Y$ by $Y^X$.

*E-mail address:* fejord@hotmail.com.

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By a *space* we mean a hereditarily Lindelöf topological space in which every open set may be written as a countable union of clopen sets. In particular, the closed sets of $X$ are exactly the zero-sets of $X$. We will say *topological space* in cases where there are no assumptions about the topology.

## 3. Hurewicz λ-sets

Let $X$ be a space. We say that $X$ is *Hurewicz* [7] provided that for every sequence $(\mathcal{O}_n)_{n\in\omega}$ of open covers of $X$, there exist finite collections $\mathcal{V}_n \subseteq \mathcal{O}_n$ such that $X = \bigcup_{n\in\omega}\bigcap_{k\geqslant n}\bigcup\mathcal{V}_k$. Since $X$ is Lindelöf, we may, without loss of generality, assume that the covers $\mathcal{O}_n$ in the definition of Hurewicz are countable.

Let $X$ be a space. We say that $X$ is a *λ-space* [9] provided that every countable subset of $X$ is a $G_\delta$-set.

Given $f, g \in \omega^\omega$ we write $f <^* g$ provided that $\{n \in \omega\colon\ f(n) \geqslant g(n)\}$ is finite. We say that $F \subseteq \omega^\omega$ is *bounded* provided that there is a $g \in \omega^\omega$ such that $f <^* g$ for every $f \in F$.

The following proposition was essentially proved by Hurewicz [7], see [14]:

**Proposition 2.** *A space $X$ has the Hurewicz property if and only if $f[X]$ is bounded for every continuous function $f : X \to \omega^\omega$.*

Let $\overline{\omega} = \omega \cup \{\infty\}$ be the one point-compactification of $\omega$. We say that $f \in \overline{\omega}^\omega$ is *eventually finite* provided that $\{n \in \omega\colon f(n) = \infty\}$ is finite. Let $\mathbb{EF} \subseteq \overline{\omega}^\omega$ be the set of all eventually finite functions. We say that $f : X \to \mathbb{EF}$ is *almost-finite* provided that $\{x \in X\colon\ \infty \in f(x)[\omega]\}$ is countable. We say that $F \subseteq \mathbb{EF}$ is *bounded* provided that there is a $g \in \omega^\omega$ such that $\{n \in \omega\colon\ f(n) \geqslant g(n)\}$ is finite for every $f \in F$. We say that $S \subseteq X$ is *co-countable* provided that $X \setminus S$ is countable.

The following theorem gives a characterization of Hurewicz λ-spaces. Its proof follows the method developed in [16].

**Theorem 3.** *Let $X$ be a space. The following conditions are equivalent*:

(a) *$F[X]$ is bounded for every continuous almost-finite function $F : X \to \mathbb{EF}$,*
(b) *$X$ is Hurewicz and $X$ is a λ-set, and*
(c) *every co-countable subset of $X$ is Hurewicz.*

The proof of Theorem 3 consists of the following three lemmas.

**Lemma 4.** *If every co-countable subset of a space $X$ is Hurewicz, then $F[X]$ is bounded in $\mathbb{EF}$ for every continuous almost-finite function $F : X \to \mathbb{EF}$.*

**Proof.** Let $F : X \to \mathbb{EF}$ be continuous and almost-finite. For each $n \in \omega$ let $G_n = \{g \in F[X]\colon\ g(k) \neq \infty$ for all $k \geqslant n\}$. Notice that $X_n = F^{-1}(G_n)$ is a co-countable set for every $n \in \omega$. So, $X_n$ is Hurewicz for every $n \in \omega$.

For each $n \in \omega$. Let $T_n : G_n \to \mathbb{EF}$ be the shift transformation $T_n(g)(k) = g(n + k)$. Since $T_n$ is continuous for every $n \in \omega$, we have, by Proposition 2, that $T_n[F[X_n]]$ is bounded in $\omega^\omega$ for each $n \in \omega$. So, $F[X_n]$ is bounded in $\mathbb{EF}$ for each $n \in \omega$. Since $X = \bigcup_{n\in\omega} X_n$, $F[X]$ is bounded in $\mathbb{EF}$. □

**Lemma 5.** *Let $X$ be a space. If $F[X]$ is bounded in $\mathbb{EF}$ for every continuous almost-finite function $F : X \to \mathbb{EF}$, then $X$ is a Hurewicz λ-space.*

**Proof.** Let $f : X \to \omega^\omega$ be continuous. Since $f$ is almost-finite, $f[X]$ is bounded in $\omega^\omega$. By Proposition 2, $X$ is Hurewicz.

We now show that $X$ is a λ-space. Let $G$ be a co-countable subset of $X$. Let $\{x_n\colon\ n \in \omega\}$ be an enumeration of $X \setminus G$.

Since $X$ is a space, closed sets are zero sets. So, we may define for each $n \in \omega$ a continuous function $f_n : X \to \overline{\omega}$ so that $f_n^{-1}(\infty) = \{x_n\}$. Define $F : X \to \mathbb{EF}$ by $F(x)(n) = f_n(x)$. Notice that $F$ is continuous and almost-finite. Thus, $F[X]$ is bounded in $\mathbb{EF}$ by some $h \in \omega^\omega$. Notice that

$$G = \bigl\{x \in X\colon\ \infty \notin F(x)[\omega]\bigr\} = F^{-1}\bigl(\bigl\{g \in \omega^\omega\colon\ g <^* h\bigr\}\bigr).$$

Notice that $\{g \in \omega^\omega\colon\ g <^* h\}$ is an $F_\sigma$-subset of $\mathbb{EF}$. It follows that $G$ is an $F_\sigma$-subset of $X$. Thus, $X$ is a λ-space. □

**Lemma 6.** *If a space $X$ is a Hurewicz λ-space, then $F[X]$ is bounded in $\mathbb{EF}$ for every continuous almost-finite function $F : X \to \mathbb{EF}$.*

**Proof.** Let $F : X \to \mathbb{EF}$ be a continuous almost-finite function. Let $G = X \setminus E$ where $E = \{x \in X\colon\ \infty \in F(x)[\omega]\}$. Since $E$ is countable, $G$ is an $F_\sigma$. Since closed subsets of Hurewicz spaces are closed and the Hurewicz property is preserved by countable unions, $G$ is Hurewicz. Thus, $F[G]$ is bounded in $\omega^\omega$. Since $F[E]$ is countable, it follows that $F[X]$ is bounded in $\mathbb{EF}$. □

## 4. Countable supremums of Fréchet filters

Let $X$ be a fixed set. A collection $\mathcal{F}$ of nonempty subsets of $X$ is a *filter* provided that $F \cap G \in \mathcal{F}$ for all $G, F \in \mathcal{F}$ and for any $F \in \mathcal{F}$ and $H$ such that $F \subseteq H$ we have $H \in \mathcal{F}$. Given a collection $\mathcal{C}$ of sets we let $\mathcal{C}^{\uparrow} = \{S\colon\ \exists C \in \mathcal{C},\ C \subseteq S\}$. Clearly, if $\mathcal{C}$ is closed under finite intersections and does not contain the empty set, then $\mathcal{C}^{\uparrow}$ is a filter. If a filter is of the form $\mathcal{F} = \mathcal{C}^{\uparrow}$ we say that $\mathcal{C}$ is a *base* for $\mathcal{F}$. A sequence $(x_n)_{n\in\omega}$ is identified with the filter $\{\{x_k\colon\ k \geqslant n\}\colon\ n \in \omega\}^{\uparrow}$ generated by its tails.

We say that two filters $\mathcal{F}$ and $\mathcal{G}$ *mesh* and write $\mathcal{F} \# \mathcal{G}$ provided that for every $F \in \mathcal{F}$ and $G \in \mathcal{G}$, $F \cap G \neq \emptyset$. Given two filters $\mathcal{F}$ and $\mathcal{G}$, we say that $\mathcal{F}$ is *finer* than $\mathcal{G}$, written $\mathcal{G} \leqslant \mathcal{F}$, provided that for every $G \in \mathcal{G}$ there is an $F \in \mathcal{F}$ such that $F \subseteq G$. Given a collection of filters $\{\mathcal{G}_\alpha\colon\ \alpha \in I\}$ we define the *supremum* of the collection $\bigvee_{\alpha\in I} \mathcal{G}_\alpha$ by $\{\bigcap_{\alpha\in J} f(\alpha)\colon\ f \in \prod_{\alpha\in I} \mathcal{G}_\alpha$ and $J \in [I]^{<\omega} \setminus \{\emptyset\}\}^{\uparrow}$, provided that each of the intersections are nonempty. In other words, $\bigvee_{\alpha\in I} \mathcal{G}_\alpha$ is the filter whose base is formed by closing the family $\bigcup_{\alpha\in I} \mathcal{G}_\alpha$ under finite intersections, provided the intersections are all nonempty. It is easy to check that if $\{\mathcal{G}_\alpha\colon\ \alpha \in I\}$ is ordered by $\leqslant$, then $\bigvee_{\alpha\in I} \mathcal{G}_\alpha$ is a filter.

If $\mathcal{F}$ has a base consisting of one (countably many) set(s), then we say that $\mathcal{F}$ is *principal* (*countably based*). We say that $\mathcal{F}$ is *Fréchet* (*strongly Fréchet* [10]) provided that for every principal (countably based) filter $\mathcal{H}$, $\mathcal{H} \# \mathcal{F}$ implies that there is a sequence $(x_n)_{n\in\omega}$ such that $(x_n)_{n\in\omega} \geqslant \mathcal{F} \vee \mathcal{H}$. The strongly Fréchet filters are also commonly known as Fréchet-$\alpha_4$ or countably bisequential filters. We say that $\mathcal{F}$ is *productively Fréchet* (see [4] and [8]) provided that for every strongly Fréchet filter $\mathcal{H}$ if $\mathcal{H} \# \mathcal{F}$, then there is a sequence $(x_n)_{n\in\omega}$ such that $(x_n)_{n\in\omega} \geqslant \mathcal{F} \vee \mathcal{H}$. It is shown in [8] that the supremum of two productively Fréchet filters is again productively Fréchet. A filter $\mathcal{F}$ is said be $\alpha_2$, see [13] and [1], provided that for every sequence $((x^n_k)_{k\in\omega})_{n\in\omega}$ of sequences such that $(x^n_k)_{k\in\omega} \geqslant \mathcal{F}$ for each $n \in \omega$ there exist a sequence $(A_n)_{n\in\omega}$ on $[\omega]^\omega$ such that $\{\bigcup_{l\leqslant n}\{x^n_k\colon\ k \in A_n\}\colon\ l \in \omega\}^{\uparrow} \geqslant \mathcal{F}$. In [12] it is shown that the product of two $\alpha_2$ filters is again $\alpha_2$. If one takes two meshing $\alpha_2$ filters $\mathcal{F}_1$ and $\mathcal{F}_2$ on $X$, then $\mathcal{F}_1 \vee \mathcal{F}_2$ can be identified with $(\mathcal{F}_1 \times \mathcal{F}_2) \vee \{(x,x)\colon\ x \in X\}$. Since the $\alpha_2$ property is stable under supremum with principal filters, it follows that $\mathcal{F}_1 \vee \mathcal{F}_2$ is $\alpha_2$.

**Theorem 7.** *Let $X$ a set and $(\mathcal{F}_n)_{n\in\omega}$ be a $\leqslant$-increasing sequence of filters on $X$. If $\mathcal{F}_n$ is $\alpha_2$ and $(x^n_k)_{k\in\omega} \geqslant \mathcal{F}_n$ for every $n \in \omega$, then there is a sequence $(B_n)_{n\in\omega}$ on $[\omega]^\omega$ such that $\{\bigcup_{l\leqslant n}\{x^n_k\colon\ k \in B_n\}\colon\ l \in \omega\}^{\uparrow} \geqslant \bigvee_{n\in\omega} \mathcal{F}_n$.*

**Proof.** Since $(x^n_k)_{k\in\omega} \geqslant \mathcal{F}_0$ for every $n$ and $\mathcal{F}_0$ is $\alpha_2$, there exists a sequence $(A^0_n)_{n\in\omega}$ on $[\omega]^\omega$ such that $\{\bigcup_{l\leqslant n}\{x^n_k\colon\ k \in A^0_n\}\colon\ l \in \omega\}^{\uparrow} \geqslant \mathcal{F}_0$. Assume that $p \geqslant 0$ and we have defined sequences $(A^0_n)^\infty_{n=0}, \ldots, (A^p_n)^\infty_{n=p}$ on $[\omega]^\omega$ such that

(a) $A^j_n \subseteq A^i_n$ for all $0 \leqslant i \leqslant j \leqslant p$ and $n \geqslant j$ and
(b) $\{\bigcup_{l\leqslant n}\{x^n_k\colon\ k \in A^i_n\}\colon\ l = i \ldots\}^{\uparrow} \geqslant \mathcal{F}_i$ for all $i \leqslant p$.

We show how to do the inductive step. Notice that $(x^n_k)_{k\in A^p_n} \geqslant \mathcal{F}_{p+1}$ for every $n \geqslant p+1$. Since $\mathcal{F}_{p+1}$ is $\alpha_2$, there is for each $n \geqslant p+1$ a $A^{p+1}_n \in [A^p_n]^\omega$ such that $\{\bigcup_{l\leqslant n}\{x^n_k\colon\ k \in A^{p+1}_n\}\colon\ l = p+1\ldots\}^{\uparrow} \geqslant \mathcal{F}_{p+1}$. This completes the induction. Let $B_n = A^n_n$ for every $n \in \omega$.

Let $l \in \omega$ and $F \in \mathcal{F}_l$. We have $\bigcup\{\{x^i_k\colon\ k \in A^l_i\}\colon\ j \leqslant i\} \subseteq F$ for some $j \geqslant l$. Let $n \geqslant j$. Now, $\{x^n_k\colon\ k \in A^n_n\} \subseteq \{x^n_k\colon\ k \in A^l_n\} \subseteq F$. Thus, $\{\bigcup_{l\leqslant n}\{x^n_k\colon\ k \in B_n\}\colon\ l \in \omega\}^{\uparrow} \geqslant \bigvee_{n\in\omega} \mathcal{F}_n$. □

**Corollary 8.** *Let $(\mathcal{F}_n)_{n\in\omega}$ be a $\leqslant$-increasing sequence of filters on $X$. If $\mathcal{F}_n$ is Fréchet and $\alpha_2$ for every $n \in \omega$, then $\bigvee_{n\in\omega} \mathcal{F}_n$ is Fréchet and $\alpha_2$.*

**Proof.** Suppose $A \subseteq X$ and $A \# (\bigvee_{n\in\omega} \mathcal{F}_n)$. For each $n \in \omega$, there is a sequence $(x^n_k)_{k\in\omega}$ on $A$ such that $(x^n_k)_{k\in\omega} \geqslant A \vee \mathcal{F}_n$. By Theorem 7, there exists a sequence $(B_n)_{n\in\omega}$ on $[\omega]^\omega$ such that $\{\bigcup_{l\leqslant n}\{x^n_k\colon\ k \in B_n\}\colon\ l \in \omega\}^{\uparrow} \geqslant \bigvee_{n\in\omega} \mathcal{F}_n$. For each $n \in \omega$ let $k_n \in B_n$. Clearly, $(x^n_{k_n})_{n\in\omega} \geqslant A \vee (\bigvee_{n\in\omega} \mathcal{F}_n)$. Thus, $\bigvee_{n\in\omega} \mathcal{F}_n$ is Fréchet.

Let $((x^n_k)_{k\in\omega})_{n\in\omega}$ be a sequence of sequences such that $(x^n_k)_{k\in\omega} \geqslant \bigvee_{n\in\omega} \mathcal{F}_n$ for all $n \in \omega$. Since $\bigvee_{n\in\omega} \mathcal{F}_n \geqslant \mathcal{F}_n$ for each $n$ there is, by Theorem 7, a sequence $(B_n)_{n\in\omega}$ on $[\omega]^\omega$ such that $\{\bigcup_{l\leqslant n}\{x^n_k\colon\ k \in B_n\}\colon\ l \in \omega\}^{\uparrow} \geqslant \bigvee_{n\in\omega} \mathcal{F}_n$. Thus, $\bigvee_{n\in\omega} \mathcal{F}_n$ is $\alpha_2$. □

**Corollary 9.** *Let $\{\mathcal{G}_n\colon\ n \in \omega\}$ be a countable collection of productively Fréchet and $\alpha_2$ filters on $X$. If $\bigvee_{n\in\omega} \mathcal{G}_n$ is defined, then $\bigvee_{n\in\omega} \mathcal{G}_n$ is productively Fréchet and $\alpha_2$.*

**Proof.** Let $\mathcal{F}_n = \bigvee^n_{i=0} \mathcal{G}_i$ for every $n \in \omega$. Now, $(\mathcal{F}_n)_{n\in\omega}$ is a $\leqslant$-increasing sequence of productively Fréchet and $\alpha_2$ filters on $X$. Notice that $\bigvee_{n\in\omega} G_n = \bigvee_{n\in\omega} \mathcal{F}_n$. By Corollary 8, $\bigvee_{n\in\omega} G_n$ is $\alpha_2$.

Suppose $\mathcal{H}$ is a strongly Fréchet filter on $X$ and $\mathcal{H} \# (\bigvee_{n\in\omega} \mathcal{F}_n)$. For each $n \in \omega$, there is a sequence $(x_k^n)_{k\in\omega} \geqslant \mathcal{H} \vee \mathcal{F}_n$. By Theorem 7, there exist a sequence $(B_n)_{n\in\omega}$ such that $\{\bigcup_{l\leqslant n}\{x_k^n\colon k \in B_n\}\colon l \in \omega\}^{\uparrow} \geqslant \bigvee_{n\in\omega} \mathcal{G}_n$. Since $\mathcal{H} \# \{\bigcup_{l\leqslant n}\{x_k^n\colon k \in B_n\}\colon l \in \omega\}^{\uparrow}$ there is a sequence $(x_n)_{n\in\omega}$ such that

$$(x_n)_{n\in\omega} \geqslant \mathcal{H} \vee \left\{\bigcup_{l\leqslant n}\{x_k^n\colon k \in B_n\}\colon l \in \omega\right\}^{\uparrow} \geqslant \mathcal{H} \vee \left(\bigvee_{n\in\omega} \mathcal{G}_n\right).$$

Thus, $\bigvee_{n\in\omega} \mathcal{F}_n$ is productively Fréchet. □

## 5. Unions of $\gamma$-type spaces

For this section $X$ will always stand for a completely regular topological space. We let $\mathcal{O}_X$ denote the collection of all open subsets of $X$. Given $W \subseteq X$ we let $\Gamma(W, \mathcal{O}_X)$ denote the filter $\{\{U \in \mathcal{O}_X\colon F \subseteq U\}\colon F \in [W]^{<\omega}\}^{\uparrow}$. Such filters used in [6] and [5].

By an *ω-cover* of $X$ we mean a collection $\mathcal{P}$ of open sets such that every finite subset of $X$ is contained in some element of $\mathcal{P}$. We say that a sequence $(P_n)_{n\in\omega}$ of open sets is a *γ-cover* of $X$ provided that for every $x \in X$ we have $x \in P_n$ for almost all $n \in \omega$. By an easy translation of the definitions we see that $\mathcal{P}$ is an $\omega$-cover of $X$ if and only if $\mathcal{P} \# \Gamma(X, \mathcal{O}_X)$. Similarly, $(P_n)_{n\in\omega}$ is an $\gamma$-cover of $X$ if and only if $(P_n)_{n\in\omega} \geqslant \Gamma(X, \mathcal{O}_X)$. We say that $X$ is a *γ-space* [3] provided that every $\omega$-cover of $X$ contains a $\gamma$-cover of $X$. After translating the definitions, the argument of [3, pp. 155–156] amounts to:

**Proposition 10.** *(See [3, pp. 155–156].) $X$ is a $\gamma$-space if and only if $\Gamma(X, \mathcal{O}_X)$ is Fréchet if and only if $\Gamma(X, \mathcal{O}_X)$ is Fréchet and $\alpha_2$.*

Following [6], we say that $X$ is a *productive γ-space* provided that $\Gamma(X, \mathcal{O}_X)$ is productively Fréchet.

Suppose $X$ and $Y$ are sets and $R \subseteq X \times Y$. For $S \subseteq X$ we let

$$RS = \{y \in Y\colon \text{there is an } x \in S \text{ such that } (x, y) \in R\}.$$

For $T \subseteq Y$ we let

$$R^{-1}T = \{x \in X\colon \text{there is an } y \in T \text{ such that } (x, y) \in R\}.$$

A class of filters $\mathbb{F}$ is said to be *$\phi_1$-composable* provided that for any two sets $X$ and $Y$, any set $A \subseteq X \times Y$, and any filter $\mathcal{F}$ on $X$ that is in $\mathbb{F}$; the filter $A\mathcal{F} = \{AF\colon F \in \mathcal{F}\}^{\uparrow}$ is again in $\mathbb{F}$ (if it is defined).

**Proposition 11.** *(See [6, Lemma 4].) Let $W \subseteq X$ and $\mathbb{F}$ be a $\phi_1$-composable class of filters. $\Gamma(W, \mathcal{O}_W)$ is in $\mathbb{F}$ if and only if $\Gamma(W, \mathcal{O}_X)$ is in $\mathbb{F}$.*

It is known that the classes of Fréchet and productive Fréchet filters are $\phi_1$-composable [8].

**Lemma 12.** *The class of Fréchet and $\alpha_2$ filters is $\phi_1$-composable.*

**Proof.** Let $X$ and $Y$ be sets, $A \subseteq X \times Y$ be a set, and $\mathcal{F}$ be a Fréchet and $\alpha_2$ filter on $X$ such that $A\mathcal{F}$ is defined. It is well known, and easy to prove, that Fréchet and $\alpha_2$ filters are strongly Fréchet. So, $\mathcal{F}$ is strongly Fréchet. Hence, $A\mathcal{F}$ is Fréchet.

Let $((y_n^k)_{n\in\omega})_{k\in\omega}$ be a sequence of sequences such that $(y_n^k)_{n\in\omega} \geqslant A\mathcal{F}$ for each $k \in \omega$.

Fix $k \in \omega$ let $\mathcal{A}_k = \{\{A^{-1}\{y_l^k\colon l > n\}\colon n \in \omega\}^{\uparrow}$. Notice that $\mathcal{A}_k \# \mathcal{F}$ and $\mathcal{A}_k$ is countably based. Since $\mathcal{F}$ is strongly Fréchet, there is a sequence $(x_n^k)_{n\in\omega} \geqslant \mathcal{A}_k \vee \mathcal{F}$. So, we may find an increasing sequence $(l_n^k)_{n\in\omega}$ on $\omega$ such that $x_n^k \in A^{-1}\{y_{l_n^k}^k\}$.

Since $(x_n^k)_{n\in\omega} \geqslant \mathcal{F}$ for every $k$, there exist a sequence $(B_k)_{k\in\omega}$ on $[\omega]^\omega$ such that $\{\bigcup_{t\leqslant k}\{x_n^k\colon n \in B_k\}\colon t \in \omega\}^{\uparrow} \geqslant \mathcal{F}$.

Let $F \in \mathcal{F}$. There is a $t \in \omega$ such that $\bigcup_{t\leqslant k}\{x_n^k\colon n \in B_k\} \subseteq F$. For $k \geqslant t$ and $n \in B_k$, we have $y_{l_n^k}^k \in A\{x_n^k\} \subseteq AF$. So, $\{\bigcup_{t\leqslant k}\{y_{l_n^k}^k\colon n \in B_k\}\colon t \in \omega\}^{\uparrow} \geqslant A\mathcal{F}$. Thus, $A\mathcal{F}$ is $\alpha_2$. □

**Lemma 13.** *Let $X$ be a topological space and $W \subseteq X$. $\Gamma(W, \mathcal{O}_W)$ is a (productive) Fréchet and $\alpha_2$ filter if and only if $\Gamma(W, \mathcal{O}_X)$ is a (productive) Fréchet and $\alpha_2$ filter.*

**Proof.** Proposition 11 and Lemma 12 together with the $\phi_1$-composability of (productive) Fréchet filters. □

We are now in a position to establish two more corollaries of Theorem 7. The first of which answers a question of Miller mentioned in the introduction.

**Corollary 14.** *Let $X$ be a space. If $(W_n)_{n\in\omega}$ is a $\subseteq$-increasing sequence of $\gamma$-spaces in $X$, then $W = \bigcup_{n\in\omega} W_n$ is $\gamma$-space.*

**Proof.** By Proposition 10 and Lemma 13, $\Gamma(W_n, \mathcal{O}_X)$ is Fréchet and $\alpha_2$ for every $n \in \omega$. Since $(W_n)_{n\in\omega}$ is $\subseteq$-increasing, $\Gamma(W_n, \mathcal{O}_X) \leqslant \Gamma(W_{n+1}, \mathcal{O}_X)$ for every $n \in \omega$. By Corollary 8, $\bigvee_{n\in\omega} \Gamma(W_n, \mathcal{O}_X)$ is Fréchet and $\alpha_2$. Finally, observe that $\Gamma(W, \mathcal{O}_X) = \bigvee_{n\in\omega} \Gamma(W_n, \mathcal{O}_X)$. Thus, $W$ is a $\gamma$-space by Proposition 10 and Lemma 13. □

The second corollary answers a question from [6].

**Corollary 15.** *Let $X$ be a space. If $\{W_n\colon n \in \omega\}$ is a countable collection of productive $\gamma$-spaces in $X$, then $W = \bigcup_{n\in\omega} W_n$ is a productive $\gamma$-space.*

**Proof.** By Proposition 10 and Lemma 13, $\Gamma(W_n, \mathcal{O}_X)$ is productively Fréchet and $\alpha_2$ for every $n \in \omega$. Observe that $\Gamma(W, \mathcal{O}_X) = \bigvee_{n\in\omega} \Gamma(W_n, \mathcal{O}_X)$. By Corollary 9, $\bigvee_{n\in\omega} \Gamma(W_n, \mathcal{O}_X)$ is productively Fréchet and $\alpha_2$. By Proposition 10 and Lemma 13, $W$ is a productive $\gamma$-space. □

## 6. Productive $\gamma$-spaces

In this section we will prove the result mentioned in the introduction and also establish some other facts about productive $\gamma$-spaces.

**Lemma 16.** *Let $X$ be a space. If $X$ is an uncountable productive $\gamma$-space, then there is a countable $Y \subseteq X$ such that $X \setminus Y$ is not a productive $\gamma$-space.*

**Proof.** Suppose $X$ is such that $X \setminus Y$ is a productive $\gamma$-space for all countable $Y \subseteq X$.

Let $\mathcal{O}$ denote the open subsets of $X$ and $\mathcal{C}$ denote the clopen subsets of $X$. Let $\mathbb{K}$ be the collection of all sequences $(O_n)_{n\in\omega}$ in $\mathcal{C}$ such that $\bigcap_{n\in\omega} \bigcup_{k\geqslant n} O_k$ is countable. Notice that $\mathbb{K}$ is closed with respect to taking subsequences. Let $\mathcal{F}$ be the filter on $\mathcal{O}$ whose base consists of sets formed by selecting a tail from each sequence in $\mathbb{K}$ and taking the union. In other words, $\mathcal{F}$ is the finest filter that is coarser than every sequence in $\mathbb{K}$. Since $X$ is a space, we may find for every $S \in [X]^{<\omega}$ a sequence $(O_n)_{n\in\omega}$ on $\mathcal{C}$ such that $\bigcap_{n\in\omega} \bigcup_{k\geqslant n} O_k = S$. So, $\mathcal{F} \# \Gamma(X, \mathcal{O})$.

Suppose $\mathcal{B}$ is a countably based filter and $\mathcal{B} \# \mathcal{F}$. We may assume that $\mathcal{B}$ has a base $\{B_k\colon k \in \omega\}$ where $B_{k+1} \subseteq B_k$ for all $k \in \omega$. For every $k \in \omega$ we have $B_k \# \mathcal{F}$. So, for every $k \in \omega$ there is a sequence $(O^k_n)_{n\in\omega} \in \mathbb{K}$ such that $B_k \# (O^k_n)_{n\in\omega}$. Since $\mathbb{K}$ is closed with respect to subsequences, we may assume that $\{O^k_n\colon n \in \omega\} \subseteq B_k$. Let $A = \bigcup_{k\in\omega} \bigcap_{n\in\omega} \bigcup_{l\geqslant n} O^k_l$. Since $A$ is countable, $X \setminus A$ is a productive $\gamma$-space. Notice that $\{X \setminus O^k_n\colon n \in \omega\}$ is an open $\gamma$-cover of $X \setminus A$ for every $k \in \omega$. In other words, $(X \setminus O^k_n)_{n\in\omega} \geqslant \Gamma(X \setminus A, \mathcal{O})$ for every $k \in \omega$. By Proposition 10 and Lemma 13, $\Gamma(X \setminus A, \mathcal{O})$ is $\alpha_2$. So, there exists a $\gamma$-cover $(X \setminus O^k_{n_k})_{k\in\omega}$ of $X \setminus A$. Now, $\bigcap_{k\in\omega} \bigcup_{l\geqslant k} O^l_{n_l} \subseteq A$. Thus, $(O^k_{n_k})_{k\in\omega} \in \mathbb{K}$. Since $\{O^k_n\colon n \in \omega\} \subseteq B_k$ for every $k$, $(O^k_{n_k})_{k\in\omega} \geqslant \mathcal{F} \vee \mathcal{B}$. Thus, $\mathcal{F}$ has the property that for any countably based filter $\mathcal{B}$ such that $\mathcal{B} \# \mathcal{F}$ there is an element of $\mathbb{K}$ finer than $\mathcal{B}$. In particular, $\mathcal{F}$ is strongly Fréchet.

Since $\mathcal{F}$ is strongly Fréchet and $\mathcal{F} \# \Gamma(X, \mathcal{O})$, there is a sequence $(O_n)_{n\in\omega} \geqslant \mathcal{F} \vee \Gamma(X, \mathcal{O})$. Since $(O_n)_{n\in\omega}$ is countably based and $(O_n)_{n\in\omega} \# \mathcal{F}$, there is a $(U_n)_{n\in\omega} \in \mathbb{K}$ such that $(U_n)_{n\in\omega} \geqslant (O_n)_{n\in\omega}$. Since $(U_n)_{n\in\omega} \geqslant \Gamma(X, \mathcal{O})$ and $\bigcap_{n\in\omega} \bigcup_{k\geqslant n} U_k$ is countable, $X$ is countable. □

**Proof of Theorem 1.** Suppose $X$ is a productive $\gamma$-space and $X \setminus Y$ is Hurewicz for every countable $Y \subseteq X$. By Theorem 3, $X$ is a $\lambda$-set.

Let $Y \subseteq X$ be countable. Since $X$ is a $\lambda$-set, $X \setminus Y$ is an $F_\sigma$-subset of $X$ for every countable $Y$. By Corollary 15 and the fact that being a productive $\gamma$-space is hereditary for closed subsets [6] we have that $X \setminus Y$ is a productive $\gamma$-space. Since $Y$ was an arbitrary countable set, we conclude that $X$ is countable, by Lemma 16. □

An immediate corollary is:

**Corollary 17.** *There is no uncountable hereditarily productive $\gamma$-space.*

The proof of Theorem 1 yields:

**Corollary 18.** *If $X \subseteq \mathbb{R}$ is a productive $\gamma$-space and a $\lambda$-set, then $X$ is countable.*

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